

Statistical theory of the many-body nuclear system

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Abstract

A recently proposed statistical theory of the mean fields associated with the ground and excited collective states of a generic many-body system is extended by increasing the dimensions of the P -space. In applying the new framework to nuclear matter, in addition to the mean field energies we obtain their fluctuations as well, together with the ones of the wavefunctions, in first order of the expansion in the complexity of the Q -space states. The physics described by the latter is assumed to be random. To extract numerical predictions out of our scheme we develop a schematic version of the approach, which, while much simplified, yields results of significance on the size of the error affecting the mean fields, on the magnitude of the residual effective interaction, on the ground state spectroscopic factor and on the mixing occurring between the vectors spanning the P -space.

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I. INTRODUCTION

We have recently addressed the problem of assessing the impact of randomness in the ground state of atomic nuclei [1–3]. In exploring this issue we assume random processes to be related to the strong repulsion the nucleons experience when they come close to each other inside the nucleus. This occurrence, while comparatively rare, is of crucial relevance for the nuclear dynamics and in the perturbative approach one tries to compute the related contribution, e. g., to the binding energy of the system, using G -matrix techniques. The latter, however, account solely for the two-body aspect of the short-range nucleon-nucleon (NN) correlations and do so only within certain approximations.

Furthermore, the large violation of the Hugenholtz-Van Hove theorem occurring in the Brueckner-Hartree-Fock (BHF) nuclear matter calculations, — not to mention the well-known failure of the BHF theory in accounting for the empirical data, — clearly demonstrates that diagrams beyond the ladder ones are needed to satisfactorily describe the physics of atomic nuclei, in particular the short-range one. Higher order diagrams in the Brueckner-Bethe-Goldstone hole-line expansion have been computed, but they still appear to fall short of accounting for the experiments.

Alternatively, one seeks for an exact solution of the nuclear many-body problem via the Green’s function Monte Carlo method. This has actually been achieved lately in light nuclei, owing to the remarkable progress in the computational capabilities of modern computers (see, e. g., Ref. [4]). Yet, even the present days number crunching power is not enough to tackle the enormously complex problem represented by medium-heavy and heavy nuclei, precisely as it happens, in a different context, for the non-perturbative QCD.

Moreover, given that these computational obstacles will eventually be surmounted, one will still have to face the question whether a non-relativistic potential model description is sufficient to encompass all the aspects of the nuclear many-body problem. Indeed, the issue of Lorentz covariance, of particular importance for the short-range physics in nuclei where large momenta are attained, is lurking in a corner, not to mention the ambiguities existing in defining and constructing a potential model description of the NN interaction at short distances, where it is so hard to disentangle particle and nuclear physics from each other and where off-shell effects become larger.

It thus appears warranted to explore alternative routes in attacking the nuclear many-body problem. Hence, we have recently examined an approach to the theory of the ground state of atomic nuclei based on the concept of averaging rather than computing most of the physics related to the strong NN collisions at short distances. As a first step we have thus set up an energy averaging procedure suitable for constructing a mean field to be eventually identified with the shell model. Here, our procedure parallels the one successfully adopted in the development of the “optical model” of nuclear reactions [5], to which it is in fact directly inspired.

Clearly, for the mean field to have a meaning an assessment should be provided for the error associated with it. Indeed, if the error turns out to be large, then the concept of mean field is no longer tenable. Hence, we have derived an expression for the error through an expansion in the complexity of the states that the NN repulsion at short distances allows to reach. At the basis of the derivation lies the hypothesis alluded to in the beginning of this Introduction, namely that the matrix elements of the NN repulsion are *random*, thus

entailing the vanishing of the average value of the error, but, of course, not of its square. It is this one that our expansion (finite in finite nuclei and fastly convergent, as we shall prove) provides.

The above outlined framework basically correspond to a statistical theory of the ground state (more generally, of the bound states) of the atomic nuclei. As a first example, we have attempted to implement it in the simplest among the latter, namely nuclear matter, assuming this system to represent, at least partially, the physics of heavy nuclei. Already in performing this task, far from trivial technical problems have been encountered, notably the one of performing the sum over the complex nuclear excited states building up the error (square of). We have been able to surmount this obstacle in Ref. [3].

It should be furthermore observed that, at the present stage, even in nuclear matter our statistical theory cannot be made parameter-free. As a consequence, its predictive power is somewhat limited: Indeed, what we actually do amounts to correlate a set of observables of nuclear matter (or heavy nuclei), either measured or computed (via, e. g., the BHF theory), — like the binding and the excited states energies and the level density, — and “predict” on this basis the mean field energies of the ground and of the collective nuclear states, — *together with the associated error*, — the residual effective interaction, which, while well defined, is presently hardly computable, and in addition the ground state spectroscopic factor.

Concerning the parameters entering in our approach, one is needed to characterize our energy averaging procedure, whereas the others are required whenever the experimental values of the above referred to observables are lacking or their theoretical evaluation not available. We shall discuss in particular the significance of the former, which plays a central role in our approach.

This work is cast in the language introduced to build the unified theory of nuclear reactions [5], at the core of which lies the partition of the nuclear Hilbert space in the P and Q sectors. We implement this partition by inserting the random aspects of the physics of the nucleus in the latter. Hence the Q -space should not be viewed as specific of a given Hamiltonian, but rather it displays universal features and indeed it is in the Q -space that the energy average is performed. The P -space embodies instead the deterministic physics of the nucleus and its dimensions should be dovetailed for this purpose. This is why the chief scope of the present research is to expand the dimensions of the P sector of the nuclear Hilbert space, thus removing a major restriction of our past work, where the P -space was limited to one dimension only.

Specifically, the enlargement of the dimensions of the P -space allows: i) to compute the fluctuations of the nuclear wave function and not only of the energy, thus assessing the error associated both with the mean field energy and its wave function; ii) to show the fast convergence of the expansion for the error, essentially stemming from the rapid increase of the nuclear level density with the excitation energy; iii) to provide an improved estimate, with respect to our past work, of the mean field energy and of the spectroscopic factor of nuclear matter; iv) to predict a quenching of an order of magnitude of the matrix elements of the NN residual effective interaction as compared to those of the bare force.

In the following we shall dwell on the above items shortly revisiting the theory in Sect. II, deepening the meaning of the energy averaging in Sect. III and broadening the P -space in Sect. IV. Next, we shall address the question of the spectroscopic factor and of the wave

function fluctuations in Sect. V. Finally, the numerical results obtained on the basis of a schematic model will be presented in Sect. VI.

II. THEORETICAL FRAMEWORK

Our statistical theory has been developed in Refs. [1–3]. For the convenience of the reader and to facilitate the understanding of the extension of the theory presented in this paper, we shortly recall in the following the basic equations characterizing our approach.

As it is well-known, the splitting of the Hilbert space induced by the projection operators P and Q transforms the Schroedinger equation

$$H\psi = E\psi \quad (2.1)$$

into the pair of coupled equations

$$(E - H_{PP})(P\psi) = H_{PQ}(Q\psi) \quad (2.2a)$$

$$(E - H_{QQ})(Q\psi) = H_{QP}(P\psi), \quad (2.2b)$$

the meaning of the symbols being obvious. From the above the equation obeyed solely by $(P\psi)$ is derived. It reads

$$\mathcal{H}(P\psi) = E(P\psi), \quad (2.3)$$

the P -space Hamiltonian being

$$\mathcal{H} = H_{PP} + H_{PQ} \frac{1}{\left(\frac{1}{e_Q}\right)^{-1} + W_{QQ}} H_{QP}, \quad (2.4)$$

with

$$e_Q = E - H_{QQ} - W_{QQ} \quad (2.5a)$$

$$W_{QQ} = H_{QP} \frac{1}{E - H_{PP}} H_{PQ}. \quad (2.5b)$$

It is of significance that although Eq. (2.3) is not an eigenvalue equation, since the energy E also appears in the denominator of its right hand side, yet its solutions only occur for those values of E which are eigenvalues of (2.1) as well; in other words, a one-to-one correspondence between the values of E allowed by (2.1) and (2.3) exists (see next section for a further discussion of this point).

Now, as already mentioned in the Introduction, in our framework the quantum deterministic aspect of nuclear dynamics is assumed to be embodied in the P -space, the chaotic one in the Q -space. Hence, the strategy of averaging over the latter follows, although, admittedly, some fuzziness does affect this partitioning. To set up the averaging procedure we start by the recognition that the wave functions in the Q -space are rapidly varying functions of the energy E , viewed as a parameter classifying their ensemble.

Accordingly, we average over this ensemble following the prescription

$$\langle f(E) \rangle = \int_{-\infty}^{\infty} dE \rho(E, \bar{E}_0, \epsilon) f(E), \quad (2.6)$$

f being a generic function to be averaged over the variable E with the distribution $\rho(E, \bar{E}_0, \epsilon)$. The latter depends, beyond E , also upon the value \bar{E}_0 around which the average, — taken over a range of E essentially set by ϵ , — is performed. A distribution convenient for our purposes is

$$\rho(E, \bar{E}_0, \epsilon) = \frac{1}{2\pi i} \frac{e^{iE\eta}}{\bar{E} - (\bar{E}_0 - \epsilon) - i\eta}, \quad (2.7)$$

which is indeed correctly normalized being

$$\int_{-\infty}^{\infty} dE \rho(E, \bar{E}_0, \epsilon) = 1 \quad (2.8)$$

(one should let $\eta \rightarrow 0^+$ after the integration has been performed). Note that Eq. (2.7) extends in some sense the Lorentz distribution of the optical model [5] to the situation of a zero width state. Hence the present formalism is especially suited to deal with the nuclear ground state, which is of course stable: We shall accordingly focus mainly on the latter in the following.

We now recall that in [1–3] the Q -space wave functions was found to read

$$(Q\psi) = \frac{1}{e_Q} H_{QP} \psi_0, \quad (2.9)$$

ψ_0 being an auxiliary function that in the end disappears from the formalism. By averaging Eq. (2.9) according to the prescriptions (2.6) and (2.7), one then finds that the energy averaged wave function of the nuclear ground state in the P -space (here denoted by the angle brackets) obeys the equation

$$\bar{\mathcal{H}} \langle P\psi \rangle = \bar{E}_0 \langle P\psi \rangle. \quad (2.10)$$

In (2.10) \bar{E}_0 is the mean field energy and

$$\bar{\mathcal{H}} = H_{PP} + H_{PQ} \frac{1}{\left\langle \frac{1}{e_Q} \right\rangle^{-1} + W_{QQ}(E = \bar{E}_0)} H_{QP} \quad (2.11)$$

is the mean field Hamiltonian. This can be further elaborated since the singularities of the operator $1/e_Q$ lie in the $\text{Im}E < 0$ half-plane [6]. Accordingly, one gets

$$\begin{aligned} \left\langle \frac{1}{e_Q} \right\rangle &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} dE \frac{e^{iE\eta}}{\bar{E} - (\bar{E}_0 - \epsilon) - i\eta} \frac{1}{E - H_{QQ} - W_{QQ}(E)} \\ &= \frac{1}{\bar{E}_0 - \epsilon - H_{QQ} - W_{QQ}(E = \bar{E}_0 - \epsilon)} \approx \frac{1}{\bar{E}_0 - \epsilon - H_{QQ} - W_{QQ}(E = \bar{E}_0)}, \end{aligned} \quad (2.12)$$

the last passage holding if the energy dependence of the operator W_{QQ} is mild and if the parameter ϵ is not too large (it should be not too small either, otherwise the energy averaging procedure becomes meaningless).

The insertion of (2.12) into (2.11) leads to the following useful alternative expression for the mean field Hamiltonian

$$\bar{\mathcal{H}} = H_{PP} + V_{PQ}V_{QP}\frac{1}{\bar{E}_0 - \epsilon - E}, \quad (2.13)$$

where the energy dependent operators

$$V_{PQ} = H_{PQ}\sqrt{\frac{\bar{E}_0 - \epsilon - E}{\bar{E}_0 - \epsilon - H_{QQ}}} \quad (2.14a)$$

and

$$V_{QP} = \sqrt{\frac{\bar{E}_0 - \epsilon - E}{\bar{E}_0 - \epsilon - H_{QQ}}}H_{QP}, \quad (2.14b)$$

represent the residual effective NN interaction. The usefulness of the Eqs. (2.13) and (2.14) was realized in Ref. [7], where it was noticed that with their help the pair of equations (2.2) can be recast, as far as $(P\psi)$ is concerned, into the form

$$(E - \bar{\mathcal{H}})(P\psi) = V_{PQ}(Q\psi) \quad (2.15a)$$

$$(E - H_{QQ})(Q\psi) = V_{QP}(P\psi), \quad (2.15b)$$

which is suitable for expressing the mean field fluctuations (the “error”).

Indeed, by exploiting the spectral decomposition of the operator $(E - \bar{\mathcal{H}})^{-1}$ in terms of the eigenfunctions ϕ_n of the mean field Hamiltonian $\bar{\mathcal{H}}$, one gets from Eq. (2.15a)

$$\begin{aligned} |P\psi\rangle &= \sum_n \frac{|\phi_n\rangle}{E - \bar{E}_n} \langle\phi_n|V_{PQ}|Q\psi\rangle \\ &= |\phi_0\rangle \frac{\langle\phi_0|V_{PQ}|Q\psi\rangle}{E - \bar{E}_0} + \left(\frac{1}{E - \bar{\mathcal{H}}}\right)' V_{PQ}|Q\psi\rangle, \end{aligned} \quad (2.16)$$

which, upon left multiplication by $\langle\phi_0|$, yields

$$\langle\phi_0|P\psi\rangle = \frac{\langle\phi_0|V_{PQ}|Q\psi\rangle}{E - \bar{E}_0}. \quad (2.17)$$

In the second term on the right hand side of Eq. (2.16), the prime stands for the omission of the $n = 0$ term in the spectral decomposition.

Next, the insertion of Eq. (2.16) into (2.15b) leads to

$$|Q\psi\rangle = \frac{1}{E - h_{QQ}}V_{QP}|\phi_0\rangle\langle\phi_0|P\psi\rangle, \quad (2.18)$$

where the operator

$$h_{QQ} = H_{QQ} + V_{QP}\left(\frac{1}{E - \bar{\mathcal{H}}}\right)'V_{PQ} \quad (2.19)$$

has been introduced. Finally, by combining (2.17) and (2.18), one arrives at the equation

$$E - \bar{E}_0 = \langle \phi_0 | V_{PQ} \frac{1}{E - h_{QQ}} V_{QP} | \phi_0 \rangle, \quad (2.20)$$

which is the basis for computing the mean field energy error.

Although Eq. (2.20) is valid for any choice of the projectors P and Q , its use is in fact appropriate when the P -space is one-dimensional, as it was indeed the case in our past work, where this choice had been made for sake of simplicity. For example, for a two-dimensional P -space, as we shall later see, one should rather single out two, rather than one, terms in the spectral decomposition of the operator $(E - \bar{\mathcal{H}})^{-1}$ on the right hand side of Eq. (2.16).

We refer the reader to Refs. [1–3] for a discussion on how the average of the square of Eq. (2.20) is actually computed (the average of (2.20) of course should vanish) and on how the complexity expansion is organized. Here, we simply remind that in the present work, — as we have done in the past, — we shall confine ourselves to the leading term of this fast converging expansion. The difference between the present results and the previous ones thus stems entirely from the enlargement of the P -space.

III. ENERGY AVERAGING

To understand better the significance of the energy averaging distribution (2.7) let us study how it works in the simple cases of a bi-dimensional (A) and of a tri-dimensional (B) Hilbert space.

A. Bi-dimensional Hilbert space

Let $|\chi_1\rangle$ and $|\chi_2\rangle$ be the two normalized states spanning the space. Here the only possible choice for the projectors clearly is

$$P \equiv |\chi_1\rangle\langle\chi_1| \quad \text{and} \quad Q \equiv |\chi_2\rangle\langle\chi_2|. \quad (3.1)$$

Then, by expanding the operator $1/(E - H_{QQ})$, Eq. (2.3) can be recast as follows

$$\left[E - |\chi_1\rangle a_{11} \langle\chi_1| - |\chi_1\rangle a_{12} \langle\chi_2| \frac{1}{E} \sum_{n=0}^{\infty} \left(\frac{a_{22}}{E} \right)^n (|\chi_2\rangle\langle\chi_2|)^n |\chi_2\rangle a_{12}^* \langle\chi_1| \right] |P\psi\rangle = 0, \quad (3.2)$$

which, upon multiplying from the left by $\langle\chi_1|$ and exploiting the idempotency of $|\chi_2\rangle\langle\chi_2|$, simplifies to

$$\left[E - a_{11} - \frac{|a_{12}|^2}{E - a_{22}} \right] \langle\chi_1|P\psi\rangle = 0, \quad (3.3)$$

where the shorthand notations

$$a_{11} = \langle\chi_1|H|\chi_1\rangle, \quad a_{22} = \langle\chi_2|H|\chi_2\rangle \quad \text{and} \quad a_{12} = \langle\chi_1|H|\chi_2\rangle \quad (3.4)$$

have been introduced. This equation is trivially solved yielding the eigenvalues

$$E_{\pm} = \frac{1}{2} \left[a_{11} + a_{22} \pm \sqrt{(a_{11} - a_{22})^2 + 4|A_{12}|^2} \right], \quad (3.5)$$

which coincide with those of H . It helps notice that the eigenvalues (3.5) are also found as intersections of the hyperbola

$$E = a_{11} + \frac{|a_{12}|^2}{\omega - a_{22}} \quad (3.6)$$

with the straight line $E = \omega$.

Also the energy averaged Hamiltonian (2.13) can be expressed in the basis spanned by χ_1 and χ_2 and one gets the mean field equation

$$\left[\bar{E} - a_{11} - \frac{|a_{12}|^2}{\bar{E} - a_{22}} \right] \langle \chi_1 | \langle P \psi \rangle \rangle = 0. \quad (3.7)$$

The latter is again trivially solved yielding

$$\bar{E}_{\pm} = \frac{1}{2} \left[a_{11} + a_{22} + \epsilon \pm \sqrt{(a_{11} - a_{22} - \epsilon)^2 + 4b^2} \right], \quad (3.8)$$

which now corresponds to the intersections of the hyperbola (3.6) (with \bar{E} replacing E) with the new straight line $\bar{E} = \omega + \epsilon$.

From Fig. 1, where the solutions E_{\pm} and \bar{E}_{\pm} are graphically displayed, it clearly appears that, while $\bar{E}_- \cong E_-$, the other solution \bar{E}_+ is much larger than E_+ , the more so the greater ϵ is. It is thus clear that the averaging distribution (2.7), while mildly affecting the eigenvalue of H lying in the P -space, drives away the one lying in the Q -space.

B. Tri-dimensional Hilbert space

The space is spanned by the normalized states $|\chi_1\rangle$, $|\chi_2\rangle$ and $|\chi_3\rangle$. Now, two choices are possible for the projectors, namely

$$P \equiv |\chi_1\rangle\langle\chi_1| + |\chi_2\rangle\langle\chi_2| \quad (3.9a)$$

$$Q \equiv |\chi_3\rangle\langle\chi_3| \quad (3.9b)$$

and

$$P \equiv |\chi_1\rangle\langle\chi_1| \quad (3.10a)$$

$$Q \equiv |\chi_2\rangle\langle\chi_2| + |\chi_3\rangle\langle\chi_3|. \quad (3.10b)$$

In both cases, Eq. (2.3) can be recast as follows

$$(E - a_{11})(E - a_{22})(E - a_{33}) - |a_{12}|^2(E - a_{33}) - |a_{13}|^2(E - a_{22}) - |a_{23}|^2(E - a_{11}) = 0, \quad (3.11)$$

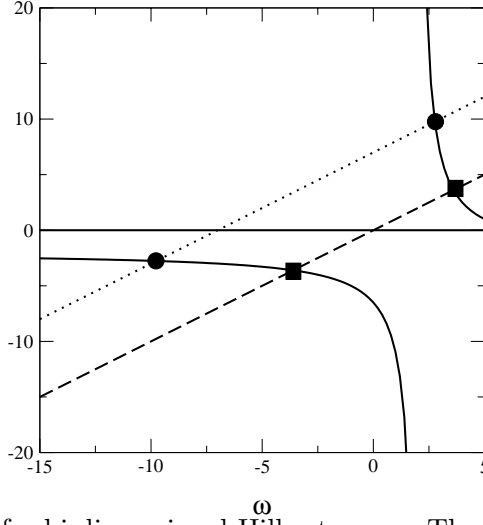


FIG. 1. The eigenvalues of a bi-dimensional Hilbert space. The matrix elements of the Hamiltonian are taken to be $a_{11} = -2$, $a_{22} = 2$ and $a_{12} = 3$, in arbitrary units. The exact eigenvalues (squares) and the ones of the energy averaged Hamiltonian $\bar{\mathcal{H}}$ (circles) are shown. They correspond to the intersections with the straight line $E = \omega + \epsilon$: The former with $\epsilon = 0$, the latter with $\epsilon = 7$. The stability of the lowest eigenvalue and the upward shift of the highest one are clearly apparent.

which is the cubic equation yielding the exact eigenvalues. Note that Eq. (3.11) is easily obtained with the choice (3.9), because in this case the operator $(E - H_{QQ})^{-1}$ is expanded in terms of the idempotent operator $|\chi_3\rangle\langle\chi_3|$. Not so with the choice (3.10), because now $(E - H_{QQ})^{-1}$ should be expanded in terms of the operator (3.10b), *which is not idempotent*. Actually, the larger the powers of the latter are, the more cumbersome they become. Yet, also in this case it can be proved that Eq. (3.11) holds valid.

Let us now examine the solutions of Eq. (2.13): As in the previous bi-dimensional case it is convenient to display the solutions graphically. For the partition (3.10), one finds that they are given by the intersections of the ϵ -independent curve

$$\bar{E} = a_{11} + \frac{|a_{12}|^2(\omega - a_{33})}{\mathcal{D}_1(\omega)} + \frac{|a_{13}|^2(\omega - a_{22})}{\mathcal{D}(\omega)} + \frac{a_{12}a_{13}^*a_{23}}{\mathcal{D}(\omega)} + \frac{a_{12}^*a_{13}a_{23}^*}{\mathcal{D}(\omega)}, \quad (3.12)$$

where

$$\mathcal{D}(\omega) = (\omega - a_{22})(\omega - a_{33}) - |a_{23}|^2, \quad (3.13)$$

with the straight line $\bar{E} = \omega + \epsilon$, as displayed in Fig. 2, where the case $\epsilon = 0$, — which clearly provides the exact eigenvalues E_i of the Schroedinger equation, — is also shown. From the figure, it transparently appears that $\bar{E}_0 \cong E_0$, whereas $\bar{E}_1 \gg E_1$ and $\bar{E}_2 \gg E_2$, the latter inequalities being stronger when the parameter ϵ is large.

In the case of the partition (3.9), the solutions are given by the intersections of the curve obtained replacing $a_{22} \rightarrow a_{22} - \epsilon$ in Eqs. (3.12) and (3.13) with the straight line $\bar{E} = \omega + \epsilon$, as displayed in Fig. 3. We face here a new situation, since now not only the straight line, but also Eqs. (3.12) and (3.13) are ϵ -dependent. Yet, one again sees that for $\epsilon = 0$ one recovers

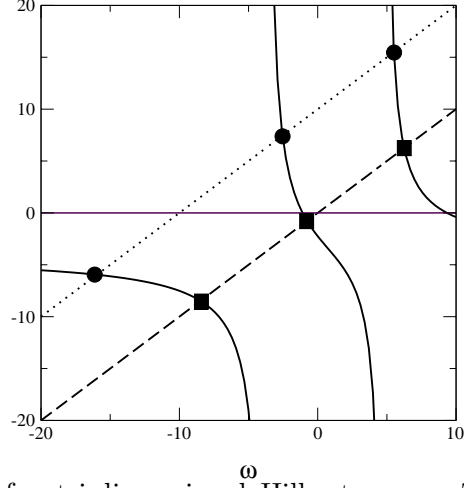


FIG. 2. The eigenvalues of a tri-dimensional Hilbert space: The case of a one-dimensional P -space. The following matrix elements of the Hamiltonian are taken: $a_{11} = -4$, $a_{22} = -2$, $a_{33} = 3$, $a_{12} = 2$, $a_{13} = -5$ and $a_{23} = 3.5$, in arbitrary units. The exact eigenvalues (squares) and the ones of the energy averaged Hamiltonian $\bar{\mathcal{H}}$ (circles) are shown. They correspond to the intersections with the straight line $E = \omega + \epsilon$: The former with $\epsilon = 0$, the latter with $\epsilon = 10$. The stability of the P -space eigenvalue and the upward shift of those belonging to the Q -space are clearly apparent.

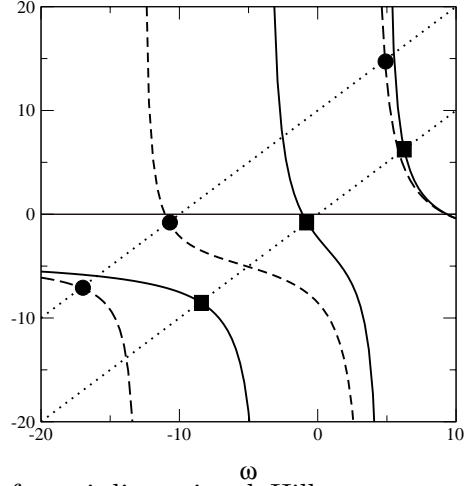


FIG. 3. The eigenvalues of a tri-dimensional Hilbert space: The case of a bi-dimensional P -space. The matrix elements of the Hamiltonian are taken as in Fig. 2. The exact eigenvalues (squares) correspond to the intersections of the straight line $E = \omega$ with the continuous curve. The eigenvalues of the energy averaged Hamiltonian $\bar{\mathcal{H}}$ correspond to the intersections of the straight line $\bar{E} = \omega + \epsilon$, with $\epsilon = 10$, with the dashed curves. Note the dependence upon ϵ . The stability of the P -space eigenvalue and the upward shift of the one belonging to the Q -space are again clearly apparent.

the eigenvalues E_i , whereas when $\epsilon \neq 0$ the intercepts occur for $\bar{E}_0 \cong E_0$ and $\bar{E}_1 \cong E_1$, but for $\bar{E}_2 \gg E_2$. Hence, we conclude that the action of the averaging distribution (2.7) affects very little the eigenvalues belonging to the P -space, while pushing off the ones in the Q -space by an amount proportional to ϵ .

IV. ENLARGING THE P -SPACE

As in our previous work, we address in the following the problem of the ground state of nuclear matter, again shortly summarizing below the equations derived in Refs. [2,3]. Indeed, the improvements brought in by the present research are more transparently appreciated by performing the analysis borning in mind the one formerly developed.

As a preliminary in carrying out our task, it is necessary to define the operators P and Q . For this purpose the natural candidates as building blocks of the P operator appear to be the eigenstates $|\phi_n\rangle$ of the mean field Hamiltonian (2.13), defined by the equation

$$\bar{\mathcal{H}}|\phi_n\rangle = \bar{E}_n|\phi_n\rangle. \quad (4.1)$$

Their finding requires, however, the solution of a difficult self-consistency problem. Hence, in Refs. [2,3] the simpler choice had been made of viewing as building blocks of P the Hartree-Fock (HF) variational solutions, which are trivial in nuclear matter.

Furthermore, there, for sake of illustration, the severe limitation of a one-dimensional P -space was adopted by setting

$$P = |\chi_{\text{HF}}\rangle\langle\chi_{\text{HF}}|, \quad (4.2)$$

$|\chi_{\text{HF}}\rangle$ being the HF ground state wave function of nuclear matter (the Fermi sphere). It was then shown that, on the basis of (4.2), one derives the mean field equation

$$\bar{E}_0 = E_{\text{HF}} + \frac{\beta^2}{\bar{E}_0 - \epsilon - E}, \quad (4.3)$$

which relates the mean field (\bar{E}_0), the HF (E_{HF}) and the true (E) energies *per particle*, and the fluctuation equation

$$E - \bar{E}_0 = \pm \frac{1}{E - \bar{\epsilon}_2} \sqrt{\frac{2}{\mathcal{N}_2}} \beta^2, \quad (4.4)$$

where

$$\beta^2 = \sum_{2\text{p-2h}} |\langle\psi_{2\text{p-2h}}|V|\chi_{\text{HF}}\rangle|^2, \quad (4.5)$$

the bras $\langle\psi_{2\text{p-2h}}|$ representing the two-particle-two-holes (2p-2h) states of nuclear matter, whose average energy *per particle* is $\bar{\epsilon}_2$. We emphasize that all the quantities appearing in Eqs. (4.3) and (4.4) are *per particle*, including the parameter ϵ and the residual effective interaction V , both of which should accordingly be thought of here as being divided by the nuclear mass number A .

Note also that Eq. (4.4) gives the “error” of the mean field energy in the first order of the complexity expansion: Hence, it is contributed to only by the sector of the Q -space set up with the 2p-2h excitations. Actually, again for sake of simplicity, the states of the Q -space, although obeying well-defined, coupled differential equations (see Ref. [2]), have been assumed to be adequately represented by the HF multi-particle–multi-hole solutions, an approximation not impairing the orthogonality constraint $P \cdot Q = 0$.

The sum in Eq. (4.5) is performed over the ensemble of the 2p-2h excited states lying in an appropriate energy range (in Refs. [2,3] taken to be fixed essentially by the parameter ϵ), whose number \mathcal{N}_2 can be computed using the Ericson’s formula [8] for the density of the spin J N -particle– N -hole nuclear states, namely

$$\rho_{ph}^{(N)}(\mathcal{E}, J) = \frac{g(g\mathcal{E})^{N-1}}{p!h!(N-1)!} \frac{2J+1}{\sqrt{8\pi}\sigma^3 N^{3/2}} \exp[-(2J+1)^2/(8N\sigma^2)], \quad (4.6)$$

where

$$g = \frac{3}{2} \frac{A}{\epsilon_F} \quad \text{and} \quad \sigma^2 = \mathcal{F} \sqrt{\frac{\mathcal{E}}{a}} \frac{1}{\hbar^2}, \quad (4.7)$$

\mathcal{F} being the nuclear moment of inertia, ϵ_F the Fermi energy, $a = A/8 \text{ MeV}^{-1}$ and \mathcal{E} the excitation energy of the system.

Now, Eqs. (4.3) and (4.4), because of the double sign appearing in the latter, set up two systems, each one including two equations, in two unknowns. Two options are possible in selecting the latter: One can choose either the ground state mean field and true energies per particle, i. e. \bar{E}_0 and E , — assuming the matrix elements of the residual effective interaction to be under control, — or \bar{E}_0 and β^2 , when E is experimentally known, — which is indeed the case in nuclear matter.

The latter is the course we followed (and continue to follow in the present work) requiring the coincidence of the two \bar{E}_0 obtained by solving the two systems separately. Actually, and notably, both systems lead to the same formal expression for the mean field energy per particle, namely

$$\bar{E}_0 = \frac{1}{2} \left\{ (E_{\text{HF}} + E + \epsilon) - \sqrt{(E_{\text{HF}} - E - \epsilon)^2 + 4\beta^2} \right\}, \quad (4.8)$$

which holds valid for $\epsilon < E_{\text{HF}} - E$, the right hand side of this inequality being positive because of the variational principle. The above, when $\beta^2 \rightarrow 0$, yields

$$\bar{E}_0 = E + \epsilon, \quad (4.9)$$

in accord with (2.14), but also with (4.4), which, for $V \rightarrow 0$, gives $\bar{E}_0 = E$, entailing that when the residual effective interaction vanishes no fluctuations occur and hence the parameter ϵ should vanish as well.

On the other hand, the two systems yield two different expressions for the sum of the matrix elements of the residual effective interaction squared, i. e.

$$\begin{aligned} \beta_l^2 = \sqrt{\mathcal{N}_2/2} \frac{E_l - \bar{\epsilon}_2}{2} \left\{ \left[E_l(1 + \sqrt{\mathcal{N}_2/2}) - (\epsilon + \sqrt{\mathcal{N}_2/2}\bar{\epsilon}_2) - E_{\text{HF}} \right] \right. \\ \left. + \sqrt{\left[E_l(1 + \sqrt{\mathcal{N}_2/2}) - (\epsilon + \sqrt{\mathcal{N}_2/2}\bar{\epsilon}_2) - E_{\text{HF}} \right]^2 + 4\epsilon(E_l - E_{\text{HF}})} \right\} \end{aligned} \quad (4.10a)$$

and

$$\beta_u^2 = \sqrt{\mathcal{N}_2/2} \frac{E_u - \bar{\epsilon}_2}{2} \left\{ \left[E_u(-1 + \sqrt{\mathcal{N}_2/2}) - (-\epsilon + \sqrt{\mathcal{N}_2/2} \bar{\epsilon}_2) + E_{\text{HF}} \right] + \sqrt{\left[E_u(-1 + \sqrt{\mathcal{N}_2/2}) - (-\epsilon + \sqrt{\mathcal{N}_2/2} \bar{\epsilon}_2) + E_{\text{HF}} \right]^2 + 4\epsilon(E_u - E_{\text{HF}})} \right\}, \quad (4.10b)$$

both of which vanish in the limit $\epsilon \rightarrow 0$, in accord with the previous discussion.

Formula (4.10a), solution of the first system of equations (the one with the “+” sign on the right hand side of (4.4)), yields the value of the *energy dependent* quantity (4.5) (here denoted by β_l^2) on the lower border of the energy band expressing the fluctuations of the ground state energy E (and thus encompassing \bar{E}_0). Formula (4.10b), solution of the second system of equations (the one with the minus sign on the right hand side of (4.4)), provides instead (4.5) (here denoted by β_u^2) on the upper border of the band (remember that $E - \bar{\epsilon}_2 < 0$).

Of course, of the energy per particle E we only know the experimental value, not the values on the borders of the band: As a consequence, we can only surmise the width W of the latter, thus providing two different inputs for the energy E appearing on the right hand side of (4.10), namely $E_u = E + W/2$ and $E_l = E - W/2$. However, we can then explore whether, for a given W , a value for the parameter ϵ can be found (not too large, not too small) such to have the two mean field energies per particle to coincide. If and when this search succeeds, then an orientation on W (or, equivalently, on the size of the fluctuations of the ground state energy) can be gained.

It is of importance to realize that the very existence of this scheme stems from the occurrence of the same quantity β^2 in both (4.3) and (4.4). This, in turn, is due to an approximation introduced in Ref. [3]: It will be further discussed in Appendix A.

We now improve upon the above approach by first letting the projector P to encompass, beyond the ground state, the 2p-2h excitations of the HF variational scheme as well. Thus, instead of Eq. (4.2), we shall write

$$P = |\chi_{\text{HF}}\rangle\langle\chi_{\text{HF}}| + \sum_{\beta} |\chi_{\text{HF}}^{2\beta}\rangle\langle\chi_{\text{HF}}^{2\beta}|, \quad (4.11)$$

where the sum is meant to be extended to the whole set of 2p-2h HF excitations $|\chi_{\text{HF}}^{2\beta}\rangle$.

With the choice (4.11) the mean field Hamiltonian (2.13) is then defined and one can compute the mean field ground state energy per particle, $\bar{E}_0 = \langle\phi_0|\bar{\mathcal{H}}|\phi_0\rangle$, using for the ket $|\phi_0\rangle$ the expression

$$|\phi_0\rangle = s_0|\chi_{\text{HF}}\rangle + \sum_{\gamma} s_2^{\gamma}|\chi_{\text{HF}}^{2\gamma}\rangle, \quad (4.12)$$

and accounting for the influence of the Q -space on the ground state mean energy per particle in first order of the complexity expansion, i. e. by setting

$$Q = \sum_{\gamma} |\chi_{\text{HF}}^{4\gamma}\rangle\langle\chi_{\text{HF}}^{4\gamma}|, \quad (4.13)$$

where the sum runs over the whole set of the HF 4p-4h excitations. In (4.12) s_0 and s_2^γ are complex coefficients, fixed, in principle, by Eq. (4.1) and satisfying the normalization condition

$$|s_0|^2 + \sum_{\gamma} |s_2^\gamma|^2 = 1. \quad (4.14)$$

After straightforward, but lengthy, algebra using (4.12) and (4.13) one finally arrives at the following new mean field equation

$$\begin{aligned} \bar{E}_0 = & |s_0|^2 E_{\text{HF}} + \sum_{\gamma} |s_2^\gamma|^2 \langle \chi_{\text{HF}}^{2\gamma} | H | \chi_{\text{HF}}^{2\gamma} \rangle + 2s_0^* \sum_{\gamma} s_2^\gamma \langle \chi_{\text{HF}} | \mathcal{V} | \chi_{\text{HF}}^{2\gamma} \rangle + \\ & + \frac{1}{\bar{E}_0 - \epsilon - E} \sum_{\beta\gamma} |s_2^\gamma|^2 |\langle \chi_{\text{HF}}^{4\beta} | V | \chi_{\text{HF}}^{2\gamma} \rangle|^2, \end{aligned} \quad (4.15)$$

\mathcal{V} being the bare NN potential.

Notably, the above equation turns out to formally coincide with (4.3). Indeed, one recognizes in the first three terms on the right hand side of Eq. (4.15) just the mean value of the original bare Hamiltonian H in the state $|\phi_0\rangle$, given in (4.12). In the thermodynamic limit characterizing nuclear matter what survives out of these pieces is just the HF energy, since the correction to the latter due to a finite number of particle-hole excitations goes to zero when divided, — in order to get the energy per particle, — by the infinite number of particles in the system: In other words, the 2p-2h admixture into (4.12) *does not change* the expectation value of the Hamiltonian.

Hence, defining

$$\zeta^2 = \sum_{\beta\gamma} |s_2^\gamma|^2 |\langle \chi_{\text{HF}}^{4\beta} | V | \chi_{\text{HF}}^{2\gamma} \rangle|^2, \quad (4.16)$$

(4.15) can be recast into the form

$$\bar{E}_0 = E_{\text{HF}} + \frac{\zeta^2}{\bar{E}_0 - \epsilon - E}, \quad (4.17)$$

whose similarity with Eq. (4.3) is transparent.

The same will take place for *any* admixture of Np - Nh HF excited states in $|\chi_{\text{HF}}\rangle$: Hence, in our framework different choices of the projection operator P lead to the same structure for the mean field equation for nuclear matter. This invariance does not hold in finite nuclei.

The only, of course important, difference between (4.5) and (4.16) is that in the latter the residual interaction V induces transitions from 2p-2h to 4p-4h states, rather than from the Fermi sphere to the 2p-2h states.

Concerning the fluctuation equation one can again use (2.20), with the state $|\phi_0\rangle$ given now by Eq. (4.12). Then invoking the randomness of the phases of the Q -space wave functions (RPA) and proceeding exactly as previously done in [2,3], one deduces the new fluctuation equation

$$E - \bar{E}_0 = \pm \frac{1}{E - \bar{\epsilon}_4} \sqrt{\frac{2}{\mathcal{N}_4}} \zeta^2, \quad (4.18)$$

where $\bar{\epsilon}_4$ denotes the average energy per particle of the 4p-4h HF states. In (4.18), \mathcal{N}_4 represents the number of 4p-4h excitations contributing to the sum over the index β in (4.16).

Hence, the “formal invariance”, with respect to the choice for the projector P , holds for both the equations at the core of our statistical approach to nuclear matter in first order of the complexity expansion.

Indeed, the inclusion of $Np-Nh$ states (with $N > 2$) into the P -space would merely imply the replacement, in (4.18), of \mathcal{N}_4 with \mathcal{N}_{N+2} and, at the same time, to have ζ^2 defined in terms of the matrix elements of V between $Np-Nh$ and $(N+2)p-(N+2)h$ states. In addition, one should of course insert in the energy denominator the average energy per particle of the $(N+2)p-(N+2)h$ HF states.

Therefore, in this connection one clearly sees that the extension of the P -space rapidly leads to the vanishing of the fluctuations, owing to the very fast increase of the number \mathcal{N}_N .

However, it should be realized that Eq. (2.20), as it stands, *is not* in general a good starting point to derive the fluctuation equation. Indeed, it selects out only *one term* in the spectral decomposition of the operator $1/(E - \bar{\mathcal{H}})$, which, as already pointed out, is justified for a one-dimensional P -space, but not so for a multidimensional one. Hence, we depart from our past work [2,3], not only by employing Eq. (4.11), but also by writing, in place of (2.16),

$$|P\psi\rangle = |\phi_0\rangle \frac{\langle\phi_0|V_{PQ}|Q\psi\rangle}{E - \bar{E}_0} + |\phi_2\rangle \frac{\langle\phi_2|V_{PQ}|Q\psi\rangle}{E - \bar{E}_2} + \left(\frac{1}{E - \bar{\mathcal{H}}}\right)'' V_{PQ}|Q\psi\rangle, \quad (4.19)$$

with an obvious meaning of the double primed operator in the last term on the right hand side. Clearly, Eq. (4.19) does not follow directly from (4.11), but it assumes that in the spectral decomposition of the $(E - \bar{\mathcal{H}})^{-1}$ operator only one prominent (collective) state $|\phi_2\rangle$ enters beyond the ground state $|\phi_0\rangle$.

Sticking to this model, instead of Eq. (2.18), we likewise use

$$|Q\psi\rangle = \frac{1}{E - h_{QQ}^{(2)}} V_{QP} \{ |\phi_0\rangle \langle\phi_0|P\psi\rangle + |\phi_2\rangle \langle\phi_2|P\psi\rangle \}, \quad (4.20)$$

being

$$h_{QQ}^{(2)} = H_{QQ} + V_{QP} \left(\frac{1}{E - \bar{\mathcal{H}}}\right)'' V_{PQ}. \quad (4.21)$$

Then, by left multiplying (4.19) (with $E = E_0$) by $\langle\phi_0|$ and using (4.20), we obtain

$$E_0 - \bar{E}_0 = \langle\phi_0|V_{PQ} \frac{1}{E_0 - h_{QQ}^{(2)}} V_{QP}|\phi_0\rangle + \langle\phi_0|V_{PQ} \frac{1}{E_0 - h_{QQ}^{(2)}} V_{QP}|\phi_2\rangle \frac{\langle\phi_2|P\psi\rangle}{\langle\phi_0|P\psi\rangle}, \quad (4.22)$$

which generalizes Eq. (2.20).

In a perfectly analogous fashion, by left multiplying (4.19) (with $E = E_2$) by $\langle\phi_2|$ and using (4.20), we obtain

$$E_2 - \bar{E}_2 = \langle\phi_2|V_{PQ} \frac{1}{E_2 - h_{QQ}^{(2)}} V_{QP}|\phi_2\rangle + \langle\phi_2|V_{PQ} \frac{1}{E_2 - h_{QQ}^{(2)}} V_{QP}|\phi_0\rangle \frac{\langle\phi_0|P\psi\rangle}{\langle\phi_2|P\psi\rangle}. \quad (4.23)$$

In the above, E_0 and E_2 stand for the first two exact eigenvalues of the Schroedinger equation; \bar{E}_0 and \bar{E}_2 for the corresponding quantities associated with Eq. (4.1). Of course, Eq. (4.23) has no counterpart in our previous work: Here, it shows that, in our frame, all the energies of the P -space fluctuate.

Now, the energy averaging of (4.22) and (4.23) vanishes by definition, but the energy averaging of their square, which yields the “error”, does not. Hence, proceeding along the lines of Refs. [2,3], we subtract on the right hand side of both equations their average values, square the expressions thus obtained and make use of RPA, keeping of our expansion in the complexity of the Q -space states the first term only. Next, we exploit the structure of $|\phi_2\rangle$, which, like $|\phi_0\rangle$, must be normalized, orthogonal to $|\phi_0\rangle$ and of the form

$$|\phi_2\rangle = \sum_{\beta} c_2^{\beta} |\chi_{\text{HF}}^{2\beta}\rangle + c_0 |\chi_{\text{HF}}\rangle. \quad (4.24)$$

We thus finally arrive at the equation

$$E_0 - \bar{E}_0 = \pm \frac{1}{E_0 - \bar{\epsilon}_4} \sqrt{\frac{2}{\mathcal{N}_4}} (\zeta^2 + r \xi^2), \quad (4.25)$$

where the further definition

$$\xi^2 = \sum_{\beta\gamma} s_2^{\gamma*} c_2^{\gamma} |\langle \chi_{\text{HF}}^{4\beta} | V | \chi_{\text{HF}}^{2\gamma} \rangle|^2 \quad (4.26)$$

has been introduced in addition to (4.16); moreover, we have set

$$r \equiv \frac{\langle \phi_2 | P \psi \rangle}{\langle \phi_0 | P \psi \rangle}. \quad (4.27)$$

Likewise, for the energy of the 2p-2h state of the P -space one obtains the fluctuation equation

$$E_2 - \bar{E}_2 = \pm \frac{1}{E_2 - \bar{\epsilon}_4} \sqrt{\frac{2}{\mathcal{N}_4}} (\eta^2 + \frac{\xi^2}{r}), \quad (4.28)$$

where, naturally,

$$\eta^2 = \sum_{\beta\gamma} |c_2^{\gamma}|^2 |\langle \chi_{\text{HF}}^{4\beta} | V | \chi_{\text{HF}}^{2\gamma} \rangle|^2. \quad (4.29)$$

We thus see that the fluctuation equations (4.25) and (4.28) are actually coupled through the term (4.26).

Concerning the mean field equations, it is clear that, in the enlarged scheme brought about by the projector (4.11) (actually, by our drastic modelling of the solutions of the mean field Hamiltonian (2.13) with P given by (4.11)) and by the $|P\psi\rangle$ given in (4.19), an equation should exist also for the energy of the 2p-2h state. It can be derived by computing $\bar{E}_2 = \langle \phi_2 | \mathcal{H} | \phi_2 \rangle$ and, notably, it turns out to read

$$\bar{E}_2 = E_{\text{HF}}^{(2)} + \frac{\eta^2}{\bar{E}_2 - \epsilon - E_2}, \quad (4.30)$$

$E_{\text{HF}}^{(2)}$ representing the HF energy per particle of the system in the 2p-2h excited state. To identify this one is not necessary, since we split $E_{\text{HF}}^{(2)}$ into a part associated with the HF ground state and a part associated with the 2p-2h *excitation energies*, both per particle. Since the latter vanishes in the thermodynamic limit, — as previously noted in commenting Eq. (4.15), then $E_{\text{HF}}^{(2)} = E_{\text{HF}}$, a relation we expect to be approximately fulfilled also in a heavy nucleus.

We conclude from the above analysis that our approach leads to a set of mean field equations, one for each of the states lying in the P -space: These equations, unlike the fluctuation ones, are not coupled.

V. NORMALIZATION AND FLUCTUATION OF THE P -SPACE GROUND STATE WAVE FUNCTION

In our framework one could identify the ground state spectroscopic factor S with the square root of the norm of $|P\psi\rangle$, the system's ground state wave function projection in P -space. To work out an explicit expression for S we commence by exploiting the completeness of the normalized eigenstates of \mathcal{H} . Hence we write

$$S^2 \equiv \langle P\psi|P\psi\rangle = \sum_{n=0}^M \langle P\psi|\phi_n\rangle \langle \phi_n|P\psi\rangle = 1 - \langle Q\psi|Q\psi\rangle. \quad (5.1)$$

Now, if we confine ourselves to set $M = 1$, as in Refs. [2,3], then the Eq. (2.18) for $|Q\psi\rangle$ is warranted and we rewrite (5.1) as follows

$$S^2 = 1 - \langle \phi_0|V_{PQ} \frac{1}{(E_0 - h_{QQ})^2} V_{QP}|\phi_0\rangle |\langle \phi_0|P\psi\rangle|^2. \quad (5.2)$$

Moreover, since the choice $M = 1$ clearly entails

$$S^2 = |\langle \phi_0|P\psi\rangle|^2, \quad (5.3)$$

by exploiting (2.20) Eq. (5.2) can be recast into the form [2,3]

$$S^2 = 1 + S^2 \left[\frac{d}{dE_0}(E_0 - \bar{E}_0) + \frac{E_0 - \bar{E}_0}{\bar{E}_0 - \epsilon - E_0} \right]. \quad (5.4)$$

Finally, employing (4.8) the expression, previously obtained in [2,3],

$$S^2 = \left[\frac{3}{2} + \frac{1}{2} \frac{E_{\text{HF}} - E_0 - \epsilon - 2d\beta^2/dE_0}{\sqrt{(E_{\text{HF}} - E_0 - \epsilon)^2 + 4\beta^2}} + \frac{\epsilon}{\bar{E}_0 - \epsilon - E_0} \right]^{-1} \quad (5.5)$$

follows, in which the energy derivative of the sum of the square moduls of the vacuum–2p-2h matrix elements of the effective interaction appears (its explicit expression is given in Ref. [3]). Note that (5.5) goes to one as $\beta^2 \rightarrow 0$, as it should.

If, however, the expression (4.20) for the Q -space wave function is used, then, rather than (5.2), one obtains for the ground state spectroscopic factor

$$\begin{aligned}
S^2 = & 1 - |\langle \phi_0 | P\psi \rangle|^2 \langle \phi_0 | V_{PQ} \frac{1}{(E_0 - h_{QQ}^{(2)})^2} V_{QP} | \phi_0 \rangle - |\langle \phi_2 | P\psi \rangle|^2 \langle \phi_2 | V_{PQ} \frac{1}{(E_0 - h_{QQ}^{(2)})^2} V_{QP} | \phi_2 \rangle \\
& - \langle \phi_0 | P\psi \rangle \langle \phi_2 | P\psi \rangle^* \langle \phi_2 | V_{PQ} \frac{1}{(E_0 - h_{QQ}^{(2)})^2} V_{QP} | \phi_0 \rangle \\
& - \langle \phi_0 | P\psi \rangle^* \langle \phi_2 | P\psi \rangle \langle \phi_0 | V_{PQ} \frac{1}{(E_0 - h_{QQ}^{(2)})^2} V_{QP} | \phi_2 \rangle.
\end{aligned} \tag{5.6}$$

The above, with the help of Eqs. (4.22) and (4.23), and assuming

$$\frac{1}{E_0 - h_{QQ}^{(2)}} \approx \frac{1}{E_2 - h_{QQ}^{(2)}}, \tag{5.7}$$

can be further elaborated and one ends up with the expression

$$S^2 = \frac{1 - \left[\frac{d(\bar{E}_2 - \bar{E}_0)}{dE_0} + \frac{\bar{E}_2 - \bar{E}_0}{E_0 - \epsilon - E_0} \right] |\langle \phi_2 | P\psi \rangle|^2}{1 + \frac{\epsilon}{E_0 - \epsilon - E_0} + \frac{d\bar{E}_0}{dE_0}}, \tag{5.8}$$

which reduces to (5.4), as it should, if $\langle \phi_2 | P\psi \rangle \rightarrow 0$, i. e. for a one-dimensional P -space.

Since, from Eq. (4.27),

$$|\langle \phi_2 | P\psi \rangle|^2 = \frac{r^2}{1 + r^2} S^2, \tag{5.9}$$

then Eq. (5.8) can be recast as follows

$$S^2 = \left\{ 1 + \frac{1}{\bar{E}_0 - \epsilon - E_0} \left[\epsilon + \frac{r^2}{1 + r^2} (\bar{E}_2 - \bar{E}_0) \right] + \frac{1}{1 + r^2} \left(\frac{d\bar{E}_0}{dE_0} + r^2 \frac{d\bar{E}_2}{dE_0} \right) \right\}^{-1}, \tag{5.10}$$

which again reduces to (5.4) as $r \rightarrow 0$.

We now address the problem of the fluctuation of $|P\psi\rangle$. For this scope, we focus on the ground state and, by combining Eqs. (2.10) and (2.15), we obtain

$$(E - \bar{\mathcal{H}})[|P\psi\rangle - |\langle P\psi \rangle\rangle] + (E - \bar{E}_0)|\langle P\psi \rangle\rangle = V_{PQ}|Q\psi\rangle \tag{5.11}$$

(the angle brackets meaning energy averaging).

Then, if use is made of the expression (2.18) for $|Q\psi\rangle$ and of the spectral decomposition of the operator $(E - \bar{\mathcal{H}})^{-1}$, one gets

$$\begin{aligned}
|P\psi\rangle - |\langle P\psi \rangle\rangle = & \sum' |\phi_n\rangle \frac{1}{E - \bar{E}_n} \langle \phi_n | V_{PQ} \frac{1}{E - h_{QQ}} V_{QP} | \phi_0 \rangle \langle \phi_0 | P\psi \rangle \\
& + \frac{1}{E - \bar{E}_0} |\phi_0\rangle \langle \phi_0 | V_{PQ} \frac{1}{E - h_{QQ}} V_{QP} | \phi_0 \rangle \langle \phi_0 | P\psi \rangle \\
& - (E - \bar{E}_0) \sum_n \frac{1}{E - \bar{E}_n} |\phi_n\rangle \langle \phi_n | \langle P\psi \rangle\rangle.
\end{aligned} \tag{5.12}$$

Now, since $|\langle P\psi \rangle| \propto |\phi_0\rangle$, from the above finally it follows

$$\begin{aligned} |P\psi\rangle - |\langle P\psi \rangle| &= (1 - |\phi_0\rangle\langle\phi_0|)^{-1} \left(\frac{1}{E - \mathcal{H}} \right)' V_{PQ} \frac{1}{E - h_{QQ}} V_{QP} |\phi_0\rangle \langle\phi_0| P\psi\rangle \\ &= \langle\phi_0| P\psi\rangle \left(\frac{1}{E_0 - \mathcal{H}} \right)' V_{PQ} \frac{1}{E_0 - h_{QQ}} V_{QP} |\phi_0\rangle, \end{aligned} \quad (5.13)$$

which *vanishes when P is given by (4.2)*, since clearly $|P\psi\rangle$ does not fluctuate in a one-dimensional P -space.

If, on the other hand, P is given by Eq. (4.11), then the above can be computed, in first order of the complexity expansion, using (4.12) for $|\phi_0\rangle$, (4.24) for $|\phi_2\rangle$ and (4.13) for Q . One ends up with the expression (we set $E = E_0$ to conform to previous notations)

$$[|P\psi\rangle - |\langle P\psi \rangle|]_1 = \langle\phi_0| P\psi\rangle \frac{|\phi_2\rangle}{E_0 - \bar{E}_2} \sum_{\beta\beta'\gamma} \langle\chi_{\text{HF}}^{2\beta}| V | \chi_{\text{HF}}^{4\gamma}\rangle \frac{c_{2\beta}^* s_{2\beta'}}{E_0 - \epsilon_{\text{HF}}^{4\gamma}} \langle\chi_{\text{HF}}^{4\gamma}| V | \chi_{\text{HF}}^{2\beta'}\rangle, \quad (5.14)$$

which can be further simplified invoking the randomness of the phases of the wave functions in the Q -space and again introducing the average energy per particle $\bar{\epsilon}_4$ for the 4p-4h HF excited states. Then, with the help of Eq. (4.26), the formula

$$[|P\psi\rangle - |\langle P\psi \rangle|]_1 = \frac{\langle\phi_0| P\psi\rangle}{(E_0 - \bar{E}_2)(E_0 - \bar{\epsilon}_4)} |\phi_2\rangle \xi^2 \quad (5.15)$$

is derived. It gives the fluctuations of the wave function in first order of the complexity expansion.

Notice that here the scalar product $\langle\phi_0| P\psi\rangle$, unlike in Refs. [2,3], does not coincide with the spectroscopic factor S , as defined in (5.1), since Eq. (5.15) has been obtained using an enlarged P -space. Rather, it measures the amount of the true ground state wave function of the system embodied in the mean field state $|\phi_0\rangle$.

VI. RESULTS FROM A SCHEMATIC MODEL

The numerical implementation of the formalism described in the previous Sections represents a challenging task.

Therefore, as a first step toward this goal, we develop here a version of our approach that, while simpler, has the merit of being amenable to numerical results.

To start with, because to solve Eq. (4.1), with the projector P given by (4.11), is a major task, we assume the two lowest solutions of the latter to be of the form

$$|\phi_0\rangle = s_0 |\chi_{\text{HF}}\rangle + s_2 |\chi_{\text{HF}}^{2\text{p-2h}}\rangle \quad (6.1a)$$

and

$$|\phi_2\rangle = -s_2 |\chi_{\text{HF}}\rangle + s_0 |\chi_{\text{HF}}^{2\text{p-2h}}\rangle, \quad (6.1b)$$

where $s_0^2 + s_2^2 = 1$. Of course, s_2 remains to be fixed.

This assumption corresponds to take in the sum on the right hand side of (4.11) one term only and places all the strength of the 2p-2h eigenstates of \mathcal{H} in a single mode. Then, by defining

$$\mu^2 = \sum_{\gamma} |\langle \chi_{\text{HF}}^{4\gamma} | V | \chi_{\text{HF}}^{2\text{p-2h}} \rangle|^2, \quad (6.2)$$

the expressions (4.16), (4.26) and (4.29) can be recast as follows

$$\zeta^2 = s_2^2 \mu^2 \quad (6.3a)$$

$$\xi^2 = s_2 \sqrt{1 - s_2^2} \mu^2 \quad (6.3b)$$

$$\eta^2 = (1 - s_2^2) \mu^2, \quad (6.3c)$$

where, beyond μ^2 , only the coefficient s_2 enters.

Accordingly, the mean field and the fluctuations equations now read

$$\bar{E}_0 = E_{\text{HF}} + \frac{s_2^2 \mu^2}{\bar{E}_0 - \epsilon - E_0} \quad (6.4a)$$

$$\bar{E}_2 = E_{\text{HF}}^{(2)} + \frac{(1 - s_2^2) \mu^2}{\bar{E}_2 - \epsilon - E_2} \quad (6.4b)$$

and

$$E_0 - \bar{E}_0 = \pm \sqrt{\frac{2}{\mathcal{N}_4}} \frac{\mu^2}{E_0 - \bar{\epsilon}_4} (s_2^2 \pm r s_2 \sqrt{1 - s_2^2}) \quad (6.5a)$$

$$E_2 - \bar{E}_2 = \pm \sqrt{\frac{2}{\mathcal{N}_4}} \frac{\mu^2}{E_2 - \bar{\epsilon}_4} (1 - s_2^2 \pm \frac{1}{r} s_2 \sqrt{1 - s_2^2}), \quad (6.5b)$$

where $E_{\text{HF}}^{(2)}$ and $\bar{\epsilon}_4$ have been previously defined. We thus see that each dimension added to the P -space entails the occurrence of *two* new systems in two equations. Furthermore, the double sign inside the round brackets on the right hand side of Eqs. (6.5) corresponds to the two options one has in choosing s_0 , namely

$$s_0 = \pm \sqrt{1 - s_2^2}. \quad (6.6)$$

Although the above equations are strictly valid in the thermodynamic limit only ($A \rightarrow \infty$), we shall make an heuristic use of them also for the nucleus $A = 208$, which is of course finite, but large.

Concerning the ground state spectroscopic factor (5.10), in the present model the energy derivative appearing in S^2 is easily found (from (6.4)) to read

$$\frac{d\bar{E}_0}{dE_0} = \frac{1}{2} + \frac{1}{2} \frac{E_{\text{HF}} - E_0 - \epsilon - 2d(s_2^2 \mu^2)/dE_0}{\sqrt{(E_{\text{HF}} - E_0 - \epsilon)^2 + 4s_2^2 \mu^2}}, \quad (6.7)$$

where account is taken of the energy dependence of the coefficient s_2 .

In accord with the general theory previously discussed, both \bar{E}_0 and \bar{E}_2 lie inside an energy band expressing the “error” they are affected by. In conformity, also the ground

state spectroscopic factor should be computed on the lower and upper borders of the ground state energy band: Hence, we need to compute on both boundaries Eq. (6.7), which in turn requires the knowledge of the energy derivative of the quantity $(s_2\mu)^2$.

For this purpose, in principle one should use the residual effective interaction on the lower boundary, which is found to be

$$s_2^2\mu_l^2 = d_0 \frac{E_0^l - \bar{\epsilon}_4}{2} \left\{ E_0^l(1 + d_0) - \epsilon - d_0\bar{\epsilon}_4 - E_{\text{HF}} \right. \\ \left. + \sqrt{[E_0^l(1 + d_0) - \epsilon - d_0\bar{\epsilon}_4 - E_{\text{HF}}]^2 + 4\epsilon(E_0^l - E_{\text{HF}})} \right\}, \quad (6.8a)$$

and on the upper one, where it reads

$$s_2^2\mu_u^2 = d_0 \frac{E_0^u - \bar{\epsilon}_4}{2} \left\{ E_0^u(-1 + d_0) + \epsilon - d_0\bar{\epsilon}_4 + E_{\text{HF}} \right. \\ \left. + \sqrt{[E_0^u(-1 + d_0) + \epsilon - d_0\bar{\epsilon}_4 + E_{\text{HF}}]^2 + 4\epsilon(E_0^u - E_{\text{HF}})} \right\}, \quad (6.8b)$$

having set

$$d_0 = \sqrt{\frac{\mathcal{N}_4}{2}} \frac{1}{1 + r\sqrt{1 - s_2^2/s_2}} \quad (6.9)$$

(the double sign stemming from (6.6)), and where, as usual, E_0^l and E_0^u are the true ground state energies on the lower and upper borders of the band: As already discussed, these are expressed in terms of the band's width W .

However, in practice, owing to the energy dependence of *both* the residual effective interaction μ and the coefficient s_2 appearing in the ground state mean field wave function, the energy derivative of (6.8) turns out to be very cumbersome to carry out. Accordingly, we evaluated numerically all the derivatives with respect to the exact ground state energy E_0 entering into the expression (5.10) for S^2 .

Note that $d_0 \rightarrow \sqrt{\mathcal{N}_4}/2$ when $r \rightarrow 0$: One thus formally recovers in this limit the results of a one-dimensional P -space [3].

In solving the systems of equations (6.4a)–(6.5a) and (6.4b)–(6.5b) we have chosen in (6.6) the plus sign. The other option need not be considered because it will provide the same results, but for the change of sign of s_2 . Indeed, our equations are clearly invariant under the simultaneous transformations $s_0 \rightarrow -s_0$ and $s_2 \rightarrow -s_2$, as it should be as the latter simply correspond to a change of phase in the states (6.1).

Concerning the other quantities needed in our equations, we have assumed for the binding energy per particle of nuclear matter

$$E_0 = -16 \text{ MeV} \quad (6.10a)$$

for the ground state and

$$E_2 = -14.8 \text{ MeV} \quad (6.10b)$$

for the 2p-2h “collective” excited state. The latter value is hinted by the calculations of the (γ, nn) and (γ, pp) photo-absorption cross-sections in nuclei [9] (n being a neutron, p a proton).

\bar{E}_0 (MeV)	\bar{E}_2 (MeV)	μ_l^2 (MeV ²)	μ_u^2 (MeV ²)	S_l	S_u	s_2	ϵ (MeV)	r
-16.001	-15.901	25	26	1.016	1.000	0.71	3.2	1.02

TABLE I. The solution of the schematic model

For the HF energies per particle we have taken

$$E_{\text{HF}} = -12 \text{ MeV} \quad (6.11a)$$

$$E_{\text{HF}}^{(2)} = -11.96 \text{ MeV} \quad (6.11b)$$

$$\bar{\epsilon}_4 = -11.92 \text{ MeV}. \quad (6.11c)$$

The first of the above actually corresponds to a typical BHF outcome (see, e. g., Ref. [10]). The other two stem from the minimal energy ($\cong 4$ MeV) of a ph excitation in a Wood-Saxon well with parameters adjusted for the nucleus Pb^{208} [11].

The choice of a minimal ph energy has been made:

- a) because the only solution of our schematic model obtains with the residual effective interaction weakened by an order of magnitude with respect to the bare one: Hence the latter can only effectively connect with the lowest 4p-4h excitations (see the results below);
- b) because only with the minimal excitation energy of the 4p-4h states Eq. (4.6) provides a number of states in an energy range set by ϵ still compatible with non-trivial fluctuation equations (6.5).

In connection with the point b) we have limited ourselves to use formula (4.6) for 4p-4h excited states with angular momentum $J = 0$. It should also be added that one can legitimately question the validity of (4.6), which is deduced in the framework of a Fermi gas by purely combinatorial methods.

With the energies per particle above quoted, — which should clearly be viewed as merely orientative but for (6.10a), — we have solved the pair of systems corresponding to Eqs. (6.4a) and (6.5a) and, as well, the one corresponding to Eqs. (6.4b) and (6.5b), fulfilling the equal mean field constraint in both cases. In addition, for sake of simplicity, we have required the nucleons' residual effective interaction μ to come out the same in both instances in accord with the schematic nature of our model.

Our findings for \bar{E}_0 , \bar{E}_2 , μ_l^2 and μ_u^2 and for the ground state spectroscopic factor are displayed in Table I. These results have been obtained for a specific choice of ϵ , which fulfills the inequalities $0 \leq \epsilon \leq E_{\text{HF}} - E = 4$ MeV, s_2 and r . Notably, in order to find a solution one has to restrict these parameters to a rather narrow range around the values quoted in the Table. Also the band width W cannot be much larger than 0.1 MeV in order to get a solution. The numbers in Table I correspond to the choice $W = 0.1$ MeV.

Although we consider the outcomes of this model as merely orientative and without any pretense of being in touch with the real physics, both because of the crudeness of the model itself and because of the uncertainties affecting our inputs, yet they are worth a few remarks.

First, the eigenstates of the mean field Hamiltonian $\bar{\mathcal{H}}$ appear to have equal projections on the two vectors spanning the bi-dimensional P -space of our model, as it follows from the values we have found for s_2 .

Second, it is clear that, with respect to our past work with a one-dimensional P -space, a reduction of the error affecting the mean field energy was to be expected. It turned out to amount to an order of magnitude. This fact points to a fast convergence of the complexity expansion or, alternatively, to a contribution of the 6p-6h excitations to the binding energy per particle (in nuclear matter) of less than 1%. This estimate is of course very rough: In fact, the magnitude of the error arises not only from the dimensions of the P -space, but as well from the proximity of the HF (or BHF) solution to the experimental value of the binding energy per particle.

Third, the massive reduction of the residual effective interaction as compared to the bare one, already found in Refs. [2,3], is here confirmed. Unfortunately, in our past work we had overlooked a factor of A (the interaction under consideration is *per particle*): The quenching there predicted was accordingly wrong, but still should be quantified in one order of magnitude (see Appendix B).

Finally, a comment is in order on the spectroscopic factor S . For this purpose we write, in terms of the states $|\chi_{\text{HF}}\rangle$ and $|\chi_{\text{HF}}^{2\text{p-2h}}\rangle$ (or, alternatively, $|\phi_0\rangle$ and $|\phi_2\rangle$) spanning our P -space,

$$\begin{aligned} |P\psi\rangle &= |\chi_{\text{HF}}\rangle\langle\chi_{\text{HF}}|\psi\rangle + |\chi_{\text{HF}}^{2\text{p-2h}}\rangle\langle\chi_{\text{HF}}^{2\text{p-2h}}|\psi\rangle \\ &= |\phi_0\rangle\langle\phi_0|P\psi\rangle + |\phi_2\rangle\langle\phi_2|P\psi\rangle. \end{aligned} \quad (6.12)$$

From the above and (6.1) it follows easily

$$\langle\phi_0|P\psi\rangle = \langle\chi_{\text{HF}}|\psi\rangle s_0 + \langle\chi_{\text{HF}}^{2\text{p-2h}}|\psi\rangle s_2 \quad (6.13)$$

$$\langle\phi_2|P\psi\rangle = -\langle\chi_{\text{HF}}|\psi\rangle s_2 + \langle\chi_{\text{HF}}^{2\text{p-2h}}|\psi\rangle s_0. \quad (6.14)$$

Inverting these equations and exploiting (4.27), we obtain

$$\langle\chi_{\text{HF}}|\psi\rangle = \langle\phi_0|P\psi\rangle s_0 - \langle\phi_2|P\psi\rangle s_2 = \langle\phi_0|P\psi\rangle (s_0 - r s_2) \quad (6.15a)$$

$$\langle\chi_{\text{HF}}^{2\text{p-2h}}|\psi\rangle = \langle\phi_0|P\psi\rangle s_2 + \langle\phi_2|P\psi\rangle s_0 = \langle\phi_0|P\psi\rangle (s_2 + r s_0). \quad (6.15b)$$

Hence, with the values for s_2 and r quoted in Table I, we conclude that

$$\langle\chi_{\text{HF}}|\psi\rangle \cong 0, \quad \langle\chi_{\text{HF}}^{2\text{p-2h}}|\psi\rangle \cong 1, \quad (6.16)$$

an outcome we are hardly ready to view as realistic.

On the other hand, taking the minus sign inside the round brackets in the fluctuation equations (6.4) and also in (6.9) (which amounts to the substitution $s_2 \rightarrow -s_2$), we get

$$\langle\chi_{\text{HF}}|\psi\rangle \cong 1, \quad \langle\chi_{\text{HF}}^{2\text{p-2h}}|\psi\rangle \cong 0, \quad (6.17)$$

which, while drastic, appears to provide a physical picture more consistent with the simple model we are employing. In fact, the above results hold because $\langle\phi_0|P\psi\rangle \cong 1$, which follows from the normalization condition

$$|\langle\phi_0|P\psi\rangle|^2 + |\langle\phi_2|P\psi\rangle|^2 = 1 \quad (6.18)$$

and from the finding that $r \cong 1$. Clearly, both (6.16) and (6.17) lead to a value of one for the spectroscopic factor S , that is the result we have found.

The two radically different solutions (6.16) and (6.17) can be reconciled through Eq. (5.15). The latter, indeed, when implemented in our schematic model, yields for the fluctuations of the ground state wave function

$$[|P\psi\rangle - |\phi_0\rangle]_1 \cong |\phi_2\rangle. \quad (6.19)$$

Likewise, by either symmetry arguments or explicit calculation, one can get for the collective 2p-2h excited state

$$[|P\psi_{2p-2h}\rangle - |\phi_2\rangle]_1 \cong |\phi_0\rangle. \quad (6.20)$$

Thus, it appears that, in the schematic model presented in this Section, while the fluctuation of the energy is puny, the fluctuation of the wave function is the largest possible. This correlation is worth being explored in more realistic models.

In conclusion, it appears warranted to view the scalar product $\langle\phi_0|P\psi\rangle$ — which is informative on how much of the true ground state wave function of the system is embodied in the corresponding mean field one, — as a better representative of S : In this last instance, we would obtain $S \cong 0.7$.

VII. CONCLUSIONS

In this paper we have first shortly revisited the statistical theory of the ground and excited collective states of general many-body systems developed in Refs. [1–3].

Next we have presented an improved version of the approach by allowing for a P -space with larger dimensions than in Refs. [2,3]. In applying this new framework to nuclear matter we have obtained both the mean field energies and the fluctuations (the “error”) of both energies and wavefunctions. This has been done in first order of the expansion in the complexity of the states of the Q -space, which, in our scheme, hosts the random aspects of the nuclear dynamics. Notably, the extension of the P -space allows to acknowledge the fast rate of convergence of the complexity expansion, due to the rapid growth of the nuclear level density with the excitation energy.

Finally, as a first step toward the testing of the theory on the physics of the atomic nuclei we have worked out a schematic model that renders the approach amenable to numerical predictions. While rough, the model has nevertheless helped us in clarifying basic aspects of our statistical approach.

In conclusion, we feel the duty to state that the statistical approach here presented germinated out of the ideas discussed by Herman Feshbach in Ref. [1]. It has been subsequently pursued by him in collaboration with us in Refs. [2,3] and in this work.

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APPENDIX A

The derivation of the coupled equations at the basis of the present approach is possible because the same quantity β^2 occurs in both (4.3) and (4.4). This occurrence stems from an approximation introduced in Ref. [3], namely

$$\begin{aligned} & \sum_{\beta\gamma} \left\langle \langle \psi_{2\beta} | V | \phi_0 \rangle^* \langle \psi_{2\gamma} | V | \phi_0 \rangle \right\rangle \left\langle \langle \psi_{2\beta} | V | \phi_0 \rangle \langle \psi_{2\gamma} | V | \phi_0 \rangle^* \right\rangle \\ & \approx 2\mathcal{A} \left\{ \sum_{\beta\gamma} \left\langle \langle \psi_{2\beta} | V | \phi_0 \rangle^* \langle \psi_{2\gamma} | V | \phi_0 \rangle \right\rangle \right\} \sum_{\beta\gamma} \left\langle \langle \psi_{2\beta} | V | \phi_0 \rangle^* \langle \psi_{2\gamma} | V | \phi_0 \rangle \right\rangle. \end{aligned} \quad (\text{A1})$$

In the above, the operator \mathcal{A} corresponds to the arithmetic mean of the quantity within the angle brackets over the set of the 2p-2h states in the energy interval ϵ centered around $\bar{\epsilon}_2$ (actually, one should take the real part of the right hand side of the previous equation: However, for the sake of the argument we shall consider real quantities).

This approximation gives reasonable results if the matrix elements are roughly of the same order of magnitude: This is the reason why the sum in (A1) is limited to an energy interval around the mean 2p-2h energy where the matrix elements are relevant.

We give here a few calculable examples that prove the above statement

$$\frac{N \sum_{n=1}^N \sin(n)^4}{2[\sum_{n=1}^N \sin(n)^2]^2} = \begin{cases} 0.725, & N = 10 \\ 0.746, & N = 100 \\ 0.750, & N = \infty \end{cases} \quad (\text{A2})$$

$$\frac{N \sum_{n=1}^N n^2}{2[\sum_{n=1}^N n]^2} = \begin{cases} 0.636, & N = 10 \\ 0.663, & N = 100 \\ 2/3, & N = \infty \end{cases} \quad (\text{A3})$$

$$\frac{N \sum_{n=1}^N \exp(-n)^2}{2[\sum_{n=1}^N \exp(-n)]^2} = \begin{cases} 2.311, & N = 10 \\ 23.11, & N = 100 \\ \infty, & N = \infty \end{cases}. \quad (\text{A4})$$

Were the approximation exact, then the above ratios should evaluate to one.

APPENDIX B

We report in the following the expression for β^2 , — the sum of the square of the vacuum \rightarrow 2p-2h matrix elements, — for the schematic interaction employed in Refs. [2,3], that is

$$\mathcal{V}(r) = g_A \frac{e^{-\mu_A r}}{r} - g_B \frac{e^{-\mu_B r}}{r} \frac{1 + P_x}{2}, \quad (\text{B1})$$

P_x being the exchange operator in coordinate space.

Following the Appendix B of Ref. [2], β^2 can be recast as the combination of direct and exchange contributions such as

$$\beta_{\text{dir}}^2 = A \frac{\varrho}{4k_F} \left\{ \frac{1}{\tilde{\mu}_A^2 - \tilde{\mu}_B^2} (\tilde{\mu}_A \arctan \tilde{\mu}_A - \tilde{\mu}_B \arctan \tilde{\mu}_B) \right. \quad (\text{B2})$$

$$+ \frac{9}{5} + \frac{17}{12}(\tilde{\mu}_A^2 + \tilde{\mu}_B^2) + \frac{1}{4}(\tilde{\mu}_A^4 + \tilde{\mu}_B^4) + \frac{1}{4}\tilde{\mu}_A^2\tilde{\mu}_B^2 \\ \left. + \frac{1}{4} \frac{1}{\tilde{\mu}_A^2 - \tilde{\mu}_B^2} \left[\tilde{\mu}_B^3(3 + \tilde{\mu}_B^2)^2 \arctan \left(\frac{1}{\tilde{\mu}_B} \right) - \tilde{\mu}_A^3(3 + \tilde{\mu}_A^2)^2 \arctan \left(\frac{1}{\tilde{\mu}_A} \right) \right] \right\} \quad (\text{B3})$$

and

$$\beta_{\text{exc}}^2 = A \frac{2}{\pi^4 \varrho} \int_0^\infty dq \frac{q}{\mu_A^2 + q^2} \int_0^\infty dr \frac{e^{-\mu_B r}}{r^6} \sin(qr) \left\{ [\sin(k_F r) - k_F r \cos(k_F r)]^2 \right. \\ \left. + [\sin(k_F r) - \sin(r\sqrt{k_F^2 - q^2}) - k_F r \cos(k_F r) + r\sqrt{k_F^2 - q^2} \cos(r\sqrt{k_F^2 - q^2})]^2 \right\}, \quad (\text{B4})$$

where $\tilde{\mu}_{A,B} = \mu_{A,B}/2k_F$ and $\varrho = 2k_F^3/3\pi^2$.

In the latter, a minor error in the expression reported in Ref. [2], essentially of no numerical consequence, has been corrected. More importantly, the necessity of normalizing β^2 to the mass number A had been overlooked in Refs. [2,3]: Hence, the previously found quenching of the residual effective interaction with respect to the bare one should also be reduced by a factor A . For example, taking ^{208}Pb as an orientation, the quenching would turn out to be of an order of magnitude.

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